JOURNAL OF APPROXIMATION THEORY 12, 99–112 (1974)

On the Variational Approach to Defining Splines on L-Shaped Regions

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Communicated by Garrett Birkhoff

1. INTRODUCTION

Let $\pi_1: a = x_0 < x_1 < \cdots < x_M = b$ be a partition of the interval $I_1 = [a, b]$. For $p > 1$, let $S^{2p}(I_1, \pi_1)$ be the set of polynomial splines of degree $2p - 1$ defined on π_1. Let $F^{(p)}[a, b] = \{f \mid f^{(p-1)}$ is abs. cont., $f^{(p)} \in L^2[a, b]\}$ and let $I_{\pi_1}^p$ denote the linear projector on $F^{(p)}[a, b]$ which associates with each $f \in F^{(p)}[a, b]$ the unique element $I_{\pi_1}^p f$ of $S^{2p}(I_1, \pi_1)$ satisfying

$$\begin{aligned} (I_{\pi_1}^p f)(x_i) &= f(x_i), \qquad i = 0,\ldots, M, \\ (I_{\pi_1}^p f)^{(j)}(x_i) &= f^{(j)}(x_i), \qquad i = 0, M; \qquad j = 0,\ldots, p-1. \end{aligned} \tag{1.1}$$

It is well known [4] that $I_{\pi_1}^p f$ uniquely minimizes

$$\int_a^b [v^{(p)}(x)]^2\, dx \tag{1.2}$$

among all $v \in F^{(p)}[a, b]$ satisfying

$$\begin{aligned} v(x_i) &= f(x_i), \qquad i = 0,\ldots, M, \\ v^{(j)}(x_i) &= f^{(j)}(x_i), \qquad i = 0, M; \qquad j = 0,\ldots, p-1. \end{aligned} \tag{1.3}$$

Let R be the rectangle $I_1 \times I_2 = [a, b] \times [c, d]$ and let $\pi = \pi_1 \times \pi_2$ be a partition of R into subrectangles. The previous result can be extended by tensor products obtaining a linear projector $I_\pi^{p,2}$ with range

$$S^{2p}(R, \pi) = S^{2p}(I_1, \pi_1) \otimes S^{2p}(I_2, \pi_2)$$

having the property that the spline interpolant determined by this projector can be characterized by variational properties.

Let L be the L-shaped region shown below and let $C^{(r,s)}(L)$ be the class of all functions f defined on L with $f^{(i,j)}$ continuous on L, $0 \leqslant i \leqslant r$, $0 \leqslant j \leqslant s$, where the superscripts indicate partial derivatives. Let $S^{2p}(L, \pi)$ denote the

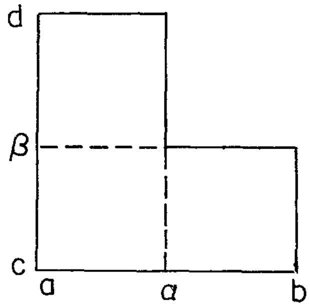

FIGURE 1

subspace of piecewise polynomials, or splines, on π which are of degree $2p - 1$ in both variables and which belong to the class $C^{(2p-2,2p-2)}(L)$. The purpose of this paper is to determine a linear interpolation projector $I^p_{\pi,L}$ onto $S^{2p}(L, \pi)$ having the property that the spline interpolant determined by this projector can be characterized by variational properties, thus providing an extension of the univariate result to L-shaped regions. This extension is achieved by noting that spline functions are representers of appropriate bounded linear functionals in an appropriate Hilbert space. It also depends in an essential way on the extension by tensor products of the univariate result to rectangles. Our construction of the projector $I^p_{\pi,L}$ provides an indication of how to proceed in the case of a more general rectangular polygon $\mathscr{R}$ to determine a spline interpolant which can be characterized by variational properties. We include an outline of this extension in Section 6.

The problem solved in this paper has also been considered in [2] and [6]. The authors of [2] were not able to find a linear interpolation projector onto $S^{2p}(L, \pi)$. The authors of [6] succeeded in finding such a projector, which is in fact simpler than the one determined here. However, the interpolation scheme determined by their projector [6, Theorem 3] is unstable in the sense that spline interpolants to a smooth function need not converge as the mesh π is successively refined. For the bicubic case the authors of [6] succeeded [6, Theorem 9] in determining a stable interpolation scheme on an L-shaped region but their scheme is *ad hoc* in the sense that it seems to provide no indication of how to proceed to determine stable interpolation schemes for higher degree splines or for other rectangular polygons.

It also would not appear that the spline interpolants defined in [6] can be characterized by variational properties. In this author's opinion, the minimization property is an important property of odd-degree interpolatory splines of one variable and should be preserved in any extension of the theory of these splines to more than one variable. For example, the minimization property for univariate splines immediately implies a best approximation property from which one can prove the uniform convergence of interpolatory splines of degree $2p - 1$ and their first $p - 1$ derivatives to moderately smooth functions. (See [1, Theorem 5.9.2].) As will be seen, the minimization properties obtained for spline interpolation on the rectangle and the L-shaped region also immediately imply best approximation properties from which one obtains analogous convergence results. (See [9, Theorem 5].)

We shall find it convenient to adopt a standard notation to use when referring to linear functionals associated with interpolation to derivative data. We shall denote by $\delta_\alpha^{(j)}$ the rule which assigns the value $f^{(j)}(\alpha)$ to the function $f \in C^{(j)}[I]$ where I is the interval of the real line indicated by the context. For $j = 0$, we shall usually omit the superscript.

2. Splines as Representers of Bounded Linear Functionals in Hilbert Space

In [5], de Boor and Lynch give general results concerning representers of bounded linear functionals in Hilbert space. We restate some of their results in the following

Lemma 1. *Let H be a real Hilbert space. Let F_i, $i = 1,..., n$, be any linearly independent set of n bounded linear functionals defined on H. Let ϕ_i be the representer of F_i, $i = 1,..., n$, i.e., $F_i f = (f, \phi_i)$ for $f \in H$, and let $S = \langle \phi_1 ,..., \phi_n \rangle$. For $f \in H$ let $P_S f$ be the orthogonal projection of f onto S. Then*

(i) *$P_S f$ is the unique element of S which interpolates f with respect to the F_i, $i = 1,..., n$;*

(ii) *of all elements $g \in H$ such that $F_i g = F_i f$, $i = 1,..., n$, $P_S f$ has the minimum norm.*

We would like to relate the variational property of odd degree interpolatory splines to Lemma 1. To do this we define an inner product $\langle \cdot, \cdot \rangle_1$ on $F^{(p)}[a, b]$ having the property that minimizing $\langle v, v \rangle_1$ over all $v \in W_f$, where W_f is the set of all $v \in F^{(p)}[a, b]$ satisfying (1.3), is equivalent to minimizing (1.2) over all $v \in W_f$. The inner product

$$\langle f, g \rangle_1 = \int_a^b f^{(p)}(x)\, g^{(p)}(x)\, dx + \sum_{j<p} f^{(j)}(a)\, g^{(j)}(a) \tag{2.1}$$

has this property, and as was shown in [5], $F^{(p)}[a, b]$ is a Hilbert space with respect to (2.1). Let S_1 be the subspace of representers with respect to (2.1) of the linear functionals

$$\Lambda_1 = \{\delta_{x_i}\}_0^M \cup \{\delta_a^{(j)}\}_1^{p-1} \cup \{\delta_b^{(j)}\}_1^{p-1}. \tag{2.2}$$

By Lemma 1, the orthogonal projection $P_{S_1}f$ of $f \in F^{(p)}[a, b]$ onto S_1 minimizes $\langle v, v\rangle_1$, or equivalently (1.2), among all $v \in W_f$. Thus $P_{S_1}f = I_{\pi_1}^p f$. This together with the fact that $\dim S^{2p}(I_1, \pi_1) = \dim S_1$ implies that $S_1 = S^{2p}(I_1, \pi_1)$, i.e., the subspace of representers S_1 is identical to the subspace of splines $S^{2p}(I_1, \pi_1)$. This observation will be important in the extension of these results to rectangular polygons.

3. Extension by Tensor Products to Rectangles

Let $R = I_1 \times I_2 = [a, b] \times [c, d]$. We define a norm on $R^{p,p} = F^{(p)}[a, b] \otimes F^{(p)}[c, d]$ and then complete $R^{p,p}$ with respect to this norm to obtain a Hilbert space. First, $F^{(p)}[c, d]$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_2 = \int_c^d f^{(p)}(y)\, g^{(p)}(y)\, dy + \sum_{j<p} f^{(j)}(c)\, g^{(j)}(c). \tag{3.1}$$

For $f \in R^{p,p}$, we have the expansion

$$\begin{aligned} f(x, y) = {} & (T_{a,p} \otimes T_{c,p})\, f(x, y) + [(1 - T_{a,p}) \otimes T_{c,p}]\, f(x, y) \\ & + [T_{a,p} \otimes (1 - T_{c,p})]\, f(x, y) + [(1 - T_{a,p}) \otimes (1 - T_{c,p})]\, f(x, y) \end{aligned} \tag{3.2}$$

where $T_{a,p}$ is the linear projector onto $\mathscr{P}_{p-1}$, the space of polynomials of degree $p - 1$ or less, defined by

$$(T_{a,p}f)(x) = \sum_{j<p} \left(f^{(j)}(a) \left(\frac{x - a}{j!}\right)^j\right) \tag{3.3}$$

and $T_{c,p}$ is defined similarly. Thus

$$(f, g) = [f, g] + \sum_{i<p} \sum_{j<p} f^{(i,j)}(a, c)\, g^{(i,j)}(a, c) \tag{3.4}$$

where

$$\begin{aligned} [f, g] = {} & \int_a^b \int_c^d f^{(p,p)}(x, y)\, g^{(p,p)}(x, y)\, dx\, dy + \sum_{j<p} \int_a^b f^{(p,j)}(x, c)\, g^{(p,j)}(x, c)\, dx \\ & + \sum_{i<p} \int_c^d f^{(i,p)}(a, y)\, g^{(i,p)}(a, y)\, dy. \end{aligned} \tag{3.5}$$

is an inner product on $R^{p,p}$. Note that $[f, f]^{1/2}$ is a seminorm with null space $\mathscr{P}_{p-1} \otimes \mathscr{P}_{p-1}$. In [8] it is shown that the completion $R_C^{p,p}$ of $R^{p,p}$ with respect to (3.4) is the class of functions with the properties:

$$\begin{aligned}
&f^{(i,j)} \in C[R], && i < p, \quad j < p,\\
&\delta_c^{(j)} f^{(p-1,0)} \text{ is abs. cont.}, && \delta_c^{(j)} f^{(0,p)} \in L^2[a,b], \quad j = 0,\dots,p-1,\\
&\delta_a^{(i)} f^{(0,p-1)} \text{ is abs. cont.}, && \delta_a^{(i)} f^{(0,p)} \in L^2[c,d], \quad i = 0,\dots,p-1,\\
&f^{(p-1,p-1)} \text{ is abs. cont.}, && f^{(p,p)} \in L^2[R].
\end{aligned}$$

For the partitions $\pi_1\colon a = x_0 < x_1 < \cdots < x_M = b$ and $\pi_2\colon c = y_0 < y_1 < \cdots < y_N = d$, we consider the set of linear functionals Λ defined by

$$\Lambda = \{\lambda \otimes \mu \mid \lambda \in \Lambda_1, \mu \in \Lambda_2\} \tag{3.6}$$

where Λ_1 is the set of linear functionals on $F^{(p)}[a, b]$ defined by (2.2) and Λ_2 is the set of linear functionals on $F^{(p)}[c, d]$ defined by

$$\Lambda_2 = \{\delta_{y_j}\}_0^N \cup \{\delta_c^{(j)}\}_1^{p-1} \cup \{\delta_d^{(j)}\}_1^{p-1}. \tag{3.7}$$

With $\pi = \pi_1 \otimes \pi_2$, let $I_\pi^{p,2} = I_{\pi_1}^p \otimes I_{\pi_2}^p$ be the linear projector onto $S^{2p}(R, \pi)$ where $I_{\pi_1}^p$ is the linear projector onto $S^{2p}(I_1, \pi_1)$ defined by (1.1) and $I_{\pi_2}^p$ is defined similarly. We remark that the spline interpolant determined by $I_\pi^{2,2}$ is just the bicubic spline of [3].

THEOREM 1. *Let $f \in R_C^{p,p}$ and let*

$$\Gamma = \{v \in R_C^{p,p} \mid \lambda v = \lambda f, \text{ all } \lambda \in \Lambda\}.$$

Then $I_\pi^{p,2} f$ uniquely minimizes $[v, v]$ among all $v \in \Gamma$.

Proof. First we note that because of the structure of the inner product (3.4) minimizing $[v, v]$ among all $v \in \Gamma$ is equivalent to minimizing (v, v) among all $v \in \Gamma$. By Lemma 1, the function that minimizes (v, v) among all $v \in \Gamma$ is the unique element in S, the subspace spanned by the representers of the set Λ, which belongs to Γ. Let $\lambda = \lambda_1 \otimes \lambda_2 \in \Lambda$ and suppose ϕ is the representer of λ with respect to (3.4), ϕ_1 is the representer of λ_1 with respect to (2.1), and ϕ_2 is the respresenter of λ_2 with respect to (3.1). Then for all $f \in R^{p,p}$

$$\lambda f = \lambda_1(\lambda_2 f) = \langle \lambda_2 f, \phi_1\rangle_1 = \langle\langle f, \phi_2\rangle_{2(y)}, \phi_1\rangle_1 = (f, \phi_1\phi_2)$$

where the subscript indicates that the inner product is taken with respect to y for fixed x. But since $R^{p,p}$ is dense in $R_C^{p,p}$, $\lambda f = (f, \phi_1\phi_2)$ holds for all $f \in R_C^{p,p}$ and $\phi = \phi_1\phi_2$.

This, together with the discussion following Lemma 1, implies that $S = S^{2p}(R, \pi)$ and the theorem is proved.

Remark. The inner product (2.1) is not the only inner product involving

$$\int_a^b f^{(p)}(x)\, g^{(p)}(x)\, dx$$

which can be defined on the linear space $F^{(p)}[a, b]$ to make it a Hilbert space. Clearly the set of functionals $\{\delta_a^{(i)}\}_0^{p-1}$ in the finite sum can be replaced by any subset $\{G_i\}_1^p$ of Λ_1, which are linearly independent over $\mathscr{P}_{p-1}$. This replacement in (2.1) has no effect on the variational problem which characterizes $I_{\pi_1}^p g$, $g \in F^{(p)}[a, b]$. It does, however, affect the variational problem of Theorem 1 in that the substitution of $\{G_i\}_1^p$ for $\{\delta_a^{(i)}\}_0^{p-1}$ results in a change in $[v, v]$. Thus $I_\pi^{p,2}f$, $f \in R_C^{p,p}$, can be characterized by a finite set of similar variational problems.

4. Extension to L-Shaped Regions

Let L be the L-shaped region of Fig. 1. Let $L^{p,p}$ be the class of functions with the properties

$$\begin{aligned}
&f^{(i,j)} \in C[L], \qquad i < p, \qquad j < p, \\
&\delta_c^{(j)} f^{(p-1,0)} \text{ is abs. cont., } \delta_c^{(j)} f^{(p,0)} \in L^2[a, b], \qquad j = 0,\dots, p-1, \\
&\delta_a^{(i)} f^{(0,p-1)} \text{ is abs. cont., } \delta_a^{(i)} f^{(0,p)} \in L^2[c, d], \qquad i = 0,\dots, p-1, \\
&f^{(p-1,p-1)} \text{ is abs. cont. on } R_i,\ f^{(p,p)} \in L^2[R_i], \qquad i = 1, 2, 3.
\end{aligned} \tag{4.1}$$

where $R_1 = [a, \alpha] \otimes [\beta, d]$, $R_2 = [a, \alpha] \otimes [c, \beta]$, $R_3 = [\alpha, b] \otimes [c, \beta]$.

We now show that $L^{p,p}$ is a Hilbert space with respect to the inner product

$$(f, g)_* = [f, g]_* + \sum_{i<p} \sum_{j<p} f^{(i,j)}(a, c)\, g^{(i,j)}(a, c) \tag{4.2}$$

where

$$[f, g]_* = \int_L\!\int f^{(p,p)}(x, y)\, g^{(p,p)}(x, y)\, dx\, dy + \sum_{j<p} \int_a^b f^{(p,j)}(x, c)\, g^{(p,j)}(x, c)\, dx$$
$$+ \sum_{i<p} \int_c^d f^{(i,p)}(a, y)\, g^{(i,p)}(a, y)\, dy. \tag{4.3}$$

With

$$T = (T_{\alpha,p} \otimes 1) + (1 \otimes T_{\beta,p}) - (T_{\alpha,p} \otimes T_{\beta,p}) \tag{4.4}$$

the "blend" of $T_{\alpha,p}$ with $T_{\beta,p}$, we define the map E on $L^{p,p}$ by

$$Ef = \begin{cases} f \text{ on } L, \\ Tf \text{ on } R\backslash L. \end{cases} \tag{4.5}$$

Since for all $f \in L^{p,p}$

$$(Tf)^{(i,0)}(\alpha, y) = f^{(i,0)}(\alpha, y), \qquad 0 \leqslant i \leqslant p-1, \qquad y \in [\beta, d],$$
$$(Tf)^{(0,j)}(x, \beta) = f^{(0,j)}(x, \beta), \qquad 0 \leqslant j \leqslant p-1, \qquad x \in [\alpha, b],$$

$Ef^{(i,j)} \in C[R]$, $i < p$, $j < p$ and E has range in $R_C^{p,p}$. Note that Tf is an example of a blended interpolant introduced by Gordon [7]. Since $(Ef)^{p,p} \equiv 0$ on $R\backslash L$, (4.2) is an inner product on $L^{p,p}$. Hence E is a linear, inner product preserving map having as a left inverse the linear and norm reducing map F given by

$$Ff = f\,|_L\,.$$

Therefore $L^{p,p}$ is a Hilbert space with respect to (4.2), $L^{p,p}$ being isomorphic to the closed subspace $E[L^{p,p}]$ of $R_C^{p,p}$.

Our objective is to find a set $\mathscr{M}$ of linear functionals with the property that the linear projector $I_{\pi,L}^p$ onto $S^{2p}(L, \pi)$ determined by $\mathscr{M}$ has the property that for $f \in L^{p,p}$, $I_{\pi,L}^p f$ uniquely minimizes $[v, v]_*$ among all $v \in L^{p,p}$ satisfying

$$\mu v = \mu f, \qquad \text{all} \quad \mu \in \mathscr{M}.$$

Here π denotes the restriction of the partition $\pi_1 \otimes \pi_2$ to L. We assume $\alpha = x_{M_1}$ and $\beta = x_{N_1}$. In [6] it is shown that

$$\dim S^p(L, \pi) = m = K + 2(p-1)(M+N+2) + 4(p-1)^2 \tag{4.6}$$

where K is the total number of mesh points. Lemma 1 shows that it is sufficient to choose m linear functionals defined on $L^{p,p}$ with the property that their representers span $S^{2p}(L, \pi)$.

Let λ be a bounded linear functional on $R_C^{p,p}$ with representer ϕ. Then λE is a linear functional on $L^{p,p}$. In addition

$$(\lambda E)f = \lambda(Ef) = (Ef, \phi) = (f, \phi\,|_L)_*\,, \qquad \text{all} \quad f \in L^{p,p}. \tag{4.7}$$

Thus λE has representer $\phi\,|_L$. This implies that the set of functionals

$$\Lambda_E = \{\lambda E \mid \lambda \in \Lambda\} \tag{4.8}$$

all have their representers in $S^{2p}(L, \pi)$. We choose $\mathscr{M}$ to be a subset of m elements of Λ_E whose representers span $S^{2p}(L, \pi)$; or what is the same thing, a subset of m elements of Λ_E which are linearly independent over $L^{p,p}$.

Let Q be the linear projector on $R_C^{p,p}$ with range $E[L^{p,p}]$ defined by

$$Q = EF.$$

As Q is also norm reducing, Q is in fact the orthogonal projector of $R_C^{p,p}$ onto $E[L^{p,p}]$. The set of linear functionals

$$\mathcal{M}_0 = \{\lambda E \mid \lambda \in \Lambda \text{ and } \lambda Q = \lambda\} \tag{4.9}$$

is clearly linearly independent over $L^{p,p}$. The set $\mathcal{M}_0$ includes the functionals associated with: (i) interpolation to values at each mesh point of L; (ii) interpolation to the first $p - 1$ normal derivatives at each boundary mesh point of L which is also on the boundary of R; and (iii) interpolation to cross-derivatives at the three corners of L which are also corners of R.

Now let Λ' be the subset of Λ containing the functionals

$$\begin{aligned} &\delta_b^{(i)} \otimes \delta_{y_j}, \qquad i = 1,..., p - 1; \qquad j = N_1 + 1,..., N, \\ &\delta_{x_i} \otimes \delta_d^{(j)}, \qquad i = M_1 + 1,..., M; \qquad j = 1,..., p - 1, \\ &\delta_b^{(i)} \otimes \delta_d^{(j)}, \qquad i = 1,..., p - 1; \qquad j = 1,..., p - 1. \end{aligned} \tag{4.10}$$

We take as the remaining functionals

$$\mathcal{M}_1 = \{\lambda E \mid \lambda \in \Lambda'\}. \tag{4.11}$$

It is straightforward to check that the set of m elements of Λ_E

$$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \tag{4.12}$$

is linearly independent over $L^{p,p}$.

This proves

THEOREM 2. *There exists a linear projector $I_{\pi,L}^p$ on $L^{p,p}$ with range $S^{2p}(L, \pi)$ which associates with each $f \in L^{p,p}$ the unique element $I_{\pi,L}^p f$ in $S^p(L, \pi)$ satisfying*

$$\mu(I_{L,\pi}^p f) = \mu f, \qquad \textit{all} \quad \mu \in \mathcal{M}$$

where $\mathcal{M}$ is the set of linear functionals defined by (4.12). *Moreover, $I_{L,\pi}^p f$ uniquely minimizes $[v, v]_*$ among all*

$$v \in \Gamma = \{v \in L^{p,p} \mid \mu v = \mu f, \textit{ all } \mu \in \mathcal{M}\}.$$

We conclude this section with the remark that if the functionals $\{\delta_a^{(i)}\}_0^{p-1}$, (or $\{\delta_c^{(j)}\}_0^{p-1}$), are replaced in (2.1) and (4.2) by the set $\{G_i\}_1^p$, $(\{\tilde{G}_j\}_1^p)$, where $\{G_i\}_1^p$, $(\{\tilde{G}_j\}_1^p)$, is any subset of p linear functionals of the set $\{\delta_a^{(i)}\}_0^{p-1} \cup \{\delta_{x_i}\}_1^M$,

$\{\delta_c^{(j)}\}_0^{p-1} \cup \{\delta_{y_j}\}_1^N$), which are linearly independent over $\mathscr{P}_{p-1}$, the methods of this section give the same linear projector $I_{\pi,L}^p$. Thus the spline interpolant $I_{\pi,L}^p f$, like $I_\pi^{p,2} f$ on the rectangle, is actually characterized by a finite set of similar variational problems.

5. Remarks on the Calculation of $I_{\pi,L}^p f$

The linear projector $I_{\pi,L}^p$ of Theorem 2 is given by $\{s_i\}_1^m$ and $\{\mu_i\}_1^m$ where $\{s_i\}_1^m$ is any basis for $S^{2p}(L, \pi)$ and $\{\mu_i\}_1^m$ is any set of m functionals which span $\mathscr{M}$, i.e., for any $f \in L^{p,p}$, $I_{\pi,L}^p f$ can be determined by solving the linear system

$$\sum_{j=1}^m \mu_i s_j = \mu_i f, \qquad i = 1,..., m.$$

The purpose of this section is to replace the complicated set of functionals $\mathscr{M}_1$ by a simpler set $\mathscr{M}_1'$ so that $\mathscr{M}$ is spanned by the set $\mathscr{M}_0 \cup \mathscr{M}_1'$. For clarity we restrict ourselves to the bicubic case ($p = 2$). Similar simplifications can be made for general p.

We first replace $(\delta_b^{(1)} \otimes \delta_{y_k})E$ by

$$\mu_k^\alpha = (\delta_b^{(1)} \otimes \delta_{y_k})E - (y_k - \beta)(\delta_b^{(1)} \otimes \delta_d^{(1)})E - (\delta_b^{(1)} \otimes \delta_\beta),$$
$$k = N_1 + 1,..., N.$$

Thus the μ_k^α are defined by

$$\mu_k^\alpha f = f^{(1,0)}(\alpha, y_k) - f^{(1,0)}(\alpha, \beta) - f^{(1,1)}(\alpha, d)(y_k - \beta),$$
$$\text{all} \quad f \in L^{p,p}, \quad k = N_1 + 1,..., N. \tag{5.1}$$

With $\bar{\mu}_{N_1+1}^\alpha = \mu_{N_1+1}^\alpha$ and $\bar{\mu}_k^\alpha = \mu_k^\alpha - \mu_{k-1}^\alpha$, $k = N_1 + 2,..., N$, we finally obtain the set $\{\bar{\mu}_k^\alpha\}_{N_1+1}^N$ where $\bar{\mu}_k^\alpha$ is given by

$$\bar{\mu}_k^\alpha f = f^{(1,0)}(\alpha, y_k) - f^{(1,0)}(\alpha, y_{k-1}) - (y_k - y_{k-1}) f^{(1,1)}(\alpha, d),$$
$$\text{all} \quad f \in L^{p,p}, \quad k = N_1 + 1,..., N. \tag{5.2}$$

Analogously we obtain the set $\{\bar{\mu}_k^\beta\}_{M_1+1}^M$ where $\bar{\mu}_k^\beta$ is given by

$$\bar{\mu}_k^\beta f = f^{(0,1)}(x_k, \beta) - f^{(0,1)}(x_{k-1}, \beta) - (x_k - x_{k-1}) f^{(1,1)}(b, \beta),$$
$$\text{all} \quad f \in L^{p,p}, \quad k = M_1 + 1,..., M. \tag{5.3}$$

We take $\mathscr{M}_1'$ to be the set

$$\mathscr{M}_1' = \{\bar{\mu}_k^\alpha\}_{N_1+1}^N \cup \{\bar{\mu}_k^\beta\}_{M_1+1}^M \cup \{\mu_0\} \tag{5.4}$$

where $\mu_0 = (\delta_b^{(1)} \otimes \delta_d^{(1)})E$ is given by

$$\mu_0 f = f^{(1,1)}(b, \beta) + f^{(1,1)}(\alpha, d) - f^{(1,1)}(\alpha, \beta), \qquad \text{all} \quad f \in L^{p,p}. \tag{5.5}$$

6. Extension to General Rectangular Polygons

Let $\mathscr{R}$ be a rectangular polygon contained in the rectangle $R = [a, b] \times [c, d]$. In this section it will be convenient to define a slightly different inner product on $R^{p,p}$ than was used in Section 3. On $R^{p,p}$ we define the inner product

$$(f, g) = [f, g] + \sum_{i=1}^{p} \sum_{j=1}^{p} f(\bar{x}_i, \bar{y}_j)\, g(\bar{x}_i, \bar{y}_j) \tag{6.1}$$

where

$$[f, g] = \int_a^b \int_c^d f^{(p,p)}(x, y)\, g^{(p,p)}(x, y)\, dx\, dy + \sum_{j=1}^{p} \int_a^b f^{(p,0)}(x, \bar{y}_j)\, g^{(p,0)}(x, \bar{y}_j)\, dx$$
$$+ \sum_{i=1}^{p} \int_c^d f^{(0,p)}(\bar{x}_i, y)\, g^{(0,p)}(\bar{x}_i, y)\, dy \tag{6.2}$$

where the $\bar{x}_i$ and $\bar{y}_j$ are chosen so that $(\bar{x}_i, \bar{y}_j) \in \mathscr{R}$, $1 \leqslant i, j \leqslant p$. We complete $R^{p,p}$ with respect to (6.1) to obtain a Hilbert space $R_C^{p,p}$ having the same properties as the Hilbert space of Section 3 except for minor changes caused by the fact that the single integrals in (6.2) are different from those in (3.5).

We define the class of functions $\mathscr{R}^{p,p}$ having properties analogous to the class of functions $L^{p,p}$ of Section 4 with minor modifications dictated by the changed inner product for $R_C^{p,p}$. The idea behind our particular choice of the class of functions $\mathscr{R}^{p,p}$ is that we will want to define an extension map E on functions in $\mathscr{R}^{p,p}$, which is similar to what was done in the case of the L-shaped region L, with the property that $E[\mathscr{R}^{p,p}] \subset R_C^{p,p}$.

Let E be the linear map on $\mathscr{R}^{p,p}$ given by

$$Ef = \begin{cases} f \text{ on } \mathscr{R} \\ Vf \text{ on } R \backslash \mathscr{R} \end{cases} \tag{6.3}$$

where V is a linear map with the property that $E[\mathscr{R}^{p,p}] \subset R_C^{p,p}$. Note that this requires

$$\frac{\partial^i (Vf)}{\partial n^i} = \frac{\partial^i f}{\partial n^i}, \qquad i = 0, \ldots, p-1,$$

on $\partial\mathscr{R} \cap \partial(R \backslash \mathscr{R})$, where $\partial/\partial n$ denotes the normal derivative. On $\mathscr{R}^{p,p}$ we define the inner product

$$(f, g)_* = (Ef, Eg). \tag{6.4}$$

We say that E defines a *minimal extension* of $\mathscr{R}^{p,p}$ if its left inverse F defined by

$$Ff = F|_{\mathscr{R}}$$

is norm reducing. The author suspects that for all rectangular polygons the minimal extension E just defined is unique. We shall give examples of minimal extensions for several regions but first we show how the idea of a minimal extension can be used to define a linear projector with range $S^{2p}(\mathscr{R}, \pi)$ where π denotes a rectangular mesh defined on R which contains the lines $x = \bar{x}_i$, $i = 1,..., p$, and $y = \bar{y}_j$, $j = 1,..., p$. We make the additional assumption that $\dim S^{2p}(\mathscr{R}, \pi) \leqslant \dim S^{2p}(R, \pi)$. See [6], Section 3 for a discussion of this restriction. Our construction parallels that for the projector $I^p_{\pi,L}$.

By construction $\mathscr{R}^{p,p}$ is a Hilbert space with respect to the inner product (6.4). Let $m = \dim S^{2p}(\mathscr{R}, \pi)$. We want to determine m linear functionals on $\mathscr{R}^{p,p}$ with the property that their representers span $S^{2p}(\mathscr{R}, \pi)$. Let E be a minimal extension. Let Q be the linear projector on $R_C^{p,p}$ with range $E[\mathscr{R}^{p,p}]$ defined by

$$Q = EF.$$

Then as F is norm reducing and E is norm preserving, Q is the orthogonal projector onto $E[\mathscr{R}^{p,p}]$. Let λ be a bounded linear functional on $R_C^{p,p}$ with representer ϕ. Then λE is a bounded linear functional on $\mathscr{R}^{p,p}$. Let $f \in \mathscr{R}^{p,p}$, then

$$\begin{aligned}\lambda Ef &= \lambda E(FE)f = \lambda QEf = (QEf, \phi)\\ &= (Ef, Q\phi) = (Ef, E(F\phi)) = (f, F\phi)_* . \end{aligned} \tag{6.5}$$

Therefore it is sufficient to choose m linearly independent functionals $\{\mu_i\}_1^m$ from the set

$$\{\lambda E \mid \lambda \in \Lambda\}$$

where Λ is the set of linear functionals on $R_C^{p,p}$ whose representers are in $S^{2p}(R, \pi)$. Applying Lemma 1 once more the linear projection $I^p_{\pi,\mathscr{R}}f$ of $f \in \mathscr{R}^{p,p}$ onto $S^{2p}(\mathscr{R}, \pi)$ determined by the $\{\mu_i\}_1^m$ uniquely minimizes $[v, v]_* = [Ev, Ev]$ among all $v \in \mathscr{R}^{p,p}$ satisfying

$$\mu_i v = \mu_i f, \qquad i = 1,..., m.$$

As an illustration we first consider the U-shaped region U of Fig. 2.

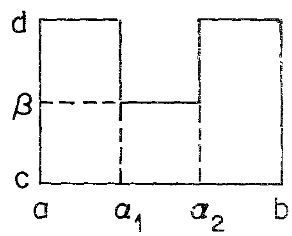

FIGURE 2

We assume that the points $\{\bar{x}_i\}_1^p$ are all contained in $[a, \alpha_1] \cup [\alpha_2, b]$ and that the points $\{\bar{y}_j\}_1^p$ are all contained in the interval $[c, \beta]$. Let V be the linear map given by

$$V = (H_{\alpha_1,\alpha_2,p} \otimes 1) + (1 \otimes T_{\beta,p}) - (H_{\alpha_1,\alpha_2,p} \otimes T_{\beta,p}) \qquad (6.6)$$

where $H_{\alpha_1,\alpha_2,p}\, g$ is the Hermite intepolating polynomial of degree $2p - 1$ interpolating $g \in C^{(p-1)}[\alpha_1, \alpha_2]$ at α_1 and α_2. By Theorem 3 of [7], Vf with V given by (6.6) minimizes

$$\int_{\alpha_1}^{\alpha_2} \int_{\beta}^{d} [v^{(p,p)}(x, y)]^2\, dx\, dy$$

among all functions $v \in R^{p,p}$ satisfying

$$\frac{\partial^i v}{\partial n^i} = \frac{\partial^i f}{\partial n^i}, \qquad i = 0,\ldots, p - 1.$$

on $\partial U \cap \partial(R \backslash U)$. Thus the left inverse F of the extension E with V given by (6.6) is norm reducing and thus E is a minimal extension of $U^{p,p}$ to $R_C^{p,p}$.

Note that if any of the $\{\bar{x}_i\}_1^p$ are chosen to be in the interval (α_1, α_2), the determination of the minimal extension is considerably more difficult since then some of the single integrals in the inner product (Ef, Ef) will depend upon Vf. In fact, the author does not know how to determine the minimal extension in this case. As an example of a rectangular polygon where this type of complication is unavoidable, consider Fig. 3.

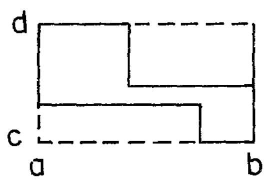

FIGURE 3

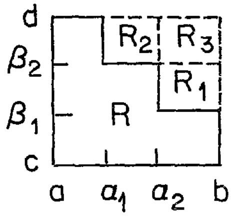

FIGURE 4

We consider one final example which illustrates the way to find the minimal extension in a case where $R/\mathscr{R}$ is not a rectangle or a disjoint set of rectangles as is the case for a T- or H-shaped region. (The determination of the minimal extension for these two regions is essentially the same as for an L or U-shaped region.) Let $\mathscr{R}$ be the rectangular polygon of Fig. 4. We assume that the points $\{\bar{x}_i\}_1^p$ are chosen to lie in the interval $[a, \alpha_1]$ and that the $\{\bar{y}_j\}_1^p$ are chosen to lie in the interval $[c, \beta_1]$. Let V be the linear map given by

$$Vf = \begin{cases} T_1 f = (T_{\alpha_2,p} \otimes 1)f + (1 \otimes T_{\beta_1,p})f - (T_{\alpha_2,p} \otimes T_{\beta_1,p})f \text{ on } R_1 , \\ T_2 f = (T_{\alpha_1,p} \otimes 1)f + (1 \otimes T_{\beta_2,p})f - (T_{\alpha_1,p} \otimes T_{\beta_2,p})f \text{ on } R_2 , \\ T_3 f = (T_{\alpha_2,p} \otimes 1)\, T_2 f + (1 \otimes T_{\beta_2,p})\, T_1 f - (T_{\alpha_2,p} \otimes T_{\beta_2,p})f \text{ on } R_3 . \end{cases} \tag{6.7}$$

Then as the extension map E with V given by (6.7) has the property that $E[\mathscr{R}^{p,p}] \subset R_C^{p,p}$ and

$$\iint_{R\setminus\mathscr{R}} [(Vf)^{(p,p)}(x, y)]^2 \, dx\, dy = 0.$$

E must be a minimal extension for the region $\mathscr{R}$ of Fig. 4.

Note that the map E is already becoming quite complicated and likewise the linear functionals used to define the linear projector $I^p_{\pi,\mathscr{R}}$. Thus the author concludes that the variational approach to defining splines is of practical value only in the case of simple rectangular polygons.

Acknowledgment

The author wishes to express her appreciation to Professor Carl de Boor for his suggestions which, she feels, have resulted in a much clearer presentation of the results of this paper.

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